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THE CALCULATION OF WALL AND FLUID TEMPERATURES FOR THE INCOMPRESSIBLE TURBULENT BOUNDARY LAYER, WITH ARBITRARY DISTRIBUTION OF WALL HEAT FLUX*

A. G. SMITH† and V. L. SHAH‡

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Abstract—Numerical solution of the partial differential equation for heat transfer in the incompressible turbulent boundary layer has been obtained for uniform $(\dot{q}''_w/\rho C_p u_1)/\surd(c_f/2)$ and for Prandtl numbers 0·7, 1 and 7. The Spalding boundary-layer velocity law was assumed, and the Schmidt method of integration used. Boundary-layer temperature distributions up to $x^+ = 10^6$ are presented, together with the "Spalding function" $St/\surd(c_f/2)$. A method is given for the application of the solutions to the case of arbitrary distribution of heat flux at the wall.

NOMENCLATURE

- a, distance along the wall at which heating starts;
- C_p, specific heat at constant pressure of the fluid;
- $c_f/2$, friction factor, $\tau_w/\rho u_1^2$;
- h, coefficient of heat transfer, $\dot{q}''_w/T_w$;
- k, thermal conductivity of the fluid;
- Pr, Prandtl number, $\rho C_p \nu/k$;
- $\dot{q}''_w$, heat flux at the surface;
- $\dot{q}''_w/[\rho u_1 C_p \surd(c_f/2)] = (\partial T/\partial \xi)_0$, wall heat flux parameter;
- St, Stanton number, $h/\rho u_1 C_p$;
- $St/\surd(c_f/2)$, Spalding function; §
- T, *difference* in temperature between a point in the boundary layer, and the mainstream;
- T_w, *difference* in temperature between wall and mainstream;
- u, velocity in boundary layer, parallel to surface;
- u_1, velocity in mainstream;
- u^+, dimensionless velocity in boundary layer, defined by equation (5);
- u_τ, friction velocity, $\surd(\tau_w/\rho)$;
- v, velocity of the fluid, normal to the wall;
- x, co-ordinate of distance along the wall;
- x^+, dimensionless distance along the wall, defined by equation (3);
- $x^+_{a,b}$, dimensionless distance between points $x = a$ and $x = b$, defined by equation (27);
- y, distance normal to the wall;
- y^+, dimensionless distance normal to the wall, defined by equation (4);
- α, thermal diffusivity, $k/\rho C_p$;
- α^+, dimensionless thermal diffusivity, defined by equation (7);
- ϵ, eddy viscosity;
- ϵ^+, dimensionless eddy viscosity, defined by equation (6);
- ϵ_h, eddy thermal diffusivity;
- θ, dimensionless temperature T/T_w;
- $\theta_{a,b,c}$, value of θ at point c for constant-heat-flux parameter between $x = a$ and $x = b$;
- ν, kinematic viscosity of the fluid;

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† University of Nottingham, England.
‡ The College of Aeronautics, Cranfield, England.
§ Note that this definition differs from that of Kestin and Persen [2] who use $St\,Pr/\surd(c_f/2)$.

ρ, density of the fluid;
ξ, dimensionless distance from the wall, defined by equation (8);
τ_w, shear stress at the wall.

INTRODUCTION

SPALDING [1] has introduced a "law of the wall" in which the dimensionless distance from the wall is given as a single function of the dimensionless velocity. Using a transformation of the energy equation due to Spalding [1], it is possible to compute temperature fields for the cases:

(1) step function of wall temperature;
(2) step function of the heat-flux parameter $\dot{q}''_w/\rho u_1 C_p \sqrt{(c_f/2)}$.

Computation of these cases then permits solution of the cases:

(1) arbitrary distribution of wall temperature;
(2) arbitrary distribution of wall heat flux.

The step function of wall temperature has been dealt with by Kestin and Persen [2], for Prandtl number unity. The step function of wall-heat-flux parameter is dealt with in the present paper for Prandtl number 0·7, 1·0 and 7. The last section of the paper shows how to compute the temperature at the wall, and within the boundary layer, for an arbitrary distribution of the heat flux at the wall.

THEORY

By use of the Von Mises transformation, Spalding [1] has reduced the energy equation for a turbulent incompressible boundary layer

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left[\alpha\frac{\partial T}{\partial y} + \epsilon_h\frac{\partial T}{\partial y}\right] \tag{1}$$

into

$$\frac{\partial T}{\partial x^+} = \frac{1}{\epsilon^+ u^+}\frac{\partial}{\partial u^+}\left[\frac{\alpha^+}{\epsilon^+}\frac{\partial T}{\partial u^+}\right] \tag{2}$$

where

$$x^+ = \int_a^x \frac{\sqrt{(\tau_w/\rho)}}{\nu}\,dx = \int_a^x \frac{u_\tau\,dx}{\nu} \tag{3}$$

$$y^+ = \frac{\sqrt{(\tau_w/\rho)}}{\nu}\,y = \frac{u_\tau y}{\nu} \tag{4}$$

$$u^+ = \frac{u}{\sqrt{(\tau_w/\rho)}} = \frac{u}{u_\tau} \text{ and } u^+ = u^+(y^+) \tag{5}$$

$$\epsilon^+ = 1 + \frac{\epsilon}{\nu} = \frac{dy^+}{du^+} \tag{6}$$

(this implies that shear stress is independent of y)

$$\alpha^+ = \frac{1}{Pr} + \frac{\epsilon_h}{\nu}. \tag{7}$$

Further, using a new variable ξ defined by

$$d\xi = \frac{\epsilon^+}{\alpha^+}\,du^+, \tag{8}$$

equation (2) can be reduced to

$$\frac{\partial T}{\partial x^+} = \frac{1}{u^+\alpha^+}\frac{\partial^2 T}{\partial \xi^2}. \tag{9}$$

Boundary conditions are

$$\left.\begin{aligned} T &= 0 \quad \text{at} \quad \xi = \infty \\ T &= 0 \quad \text{at} \quad x^+ \leqslant 0. \end{aligned}\right\} \tag{10}$$

An additional boundary condition in the present solution is

$$\left(\frac{\partial T}{\partial \xi}\right)_{\xi=0} = \text{const.} \tag{11}$$

Spalding [1] has derived a single expression for the distribution of the velocity and in the universal turbulent boundary layer in the form of $y^+(u^+)$ instead of the usual form $u^+(y^+)$. It is

$$y^+ = u^+ + \frac{1}{E}\left[e^{Ku^+} - 1 - Ku^+ - \frac{(Ku^+)^2}{2!} - \frac{(Ku^+)^3}{3!} - \frac{(Ku^+)^4}{4!}\right]. \tag{12}$$

with $K = 0{\cdot}407$ and $1/E = 0{\cdot}0991$.

From equation (12) an expression for ϵ^+ can be obtained, using equation (6). This gives

$$\epsilon^+ = 1 + \frac{K}{E}\left[e^{Ku^+} - 1 - Ku^+ - \frac{(Ku^+)^2}{2!} - \frac{(Ku^+)^3}{3!}\right]. \tag{13}$$

A further assumption that the turbulent Prandtl number is unity ($\epsilon_h = \epsilon$) leads to an expression for a^+

$$a^+ = \frac{1}{Pr} + \frac{K}{E}\left\{e^{Ku^+} - 1 - Ku^+ - \frac{(Ku^+)^2}{2!} - \frac{(Ku^+)^3}{3!}\right\}. \quad (14)$$

Therefore the quantity u^+a^+, for various Prandtl numbers, can be obtained as a function of ξ. This reduces equation (9) to

$$\frac{\partial T}{\partial x^+} = \frac{1}{f(\xi)}\frac{\partial^2 T}{\partial \xi^2}. \quad (15)$$

The values of $f(\xi)$ for $Pr = 0{\cdot}7$, 1 and 7 are shown in Fig. 1.

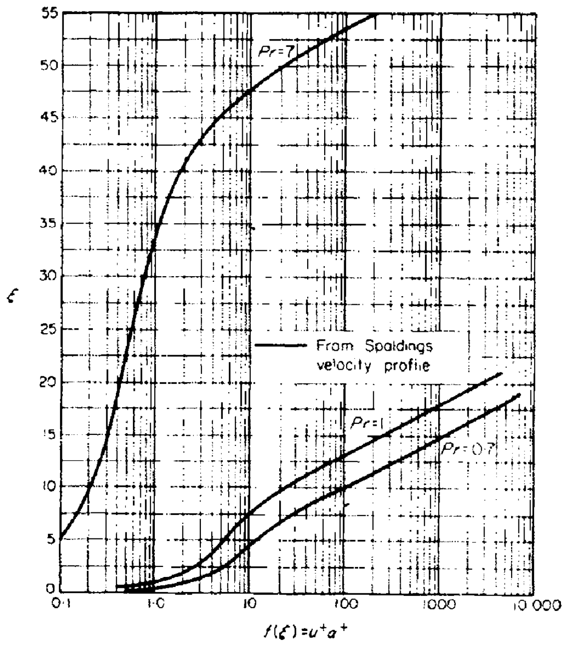

FIG. 1. Values of $u^+a^+(\xi)$.

For the case of uniform surface temperature, various investigators have solved equation (15). Spalding [1] has solved it approximately for $Pr = 1$, by use of the energy integral equation and by assuming a temperature profile in the boundary layer. Muralidharan [3] has solved it on an analogue computer for $Pr = 0{\cdot}7$, 1 and 7. Kestin and Persen [2] have solved it on a digital computer for $Pr = 1$.

In the present paper, a solution of equation (15) has been made for uniform $\partial T/\partial\xi$ at the wall and for $Pr = 0{\cdot}7$, 1 and 7.

NUMERICAL SOLUTION

It will be seen that equation (15) is very similar to the heat-conduction equation, and may be solved by the finite difference method of E. Schmidt.

The equation (15) can be written in finite-difference form as

$$T_{(x^++\Delta x^+,\xi)} - T_{(x^+,\xi)} = \frac{\Delta x^+}{f(\xi)(\Delta\xi)^2}\left[T_{(x^+,\xi+\Delta\xi)} + T_{(x^+,\xi-\Delta\xi)} - 2T_{(x^+,\xi)}\right]. \quad (16)$$

From equation (16), $T_{(x^++\Delta x^+)}$ can be calculated from $T_{(x^+)}$ which in turn can be calculated from $T_{(x^+-\Delta x^+)}$, and so on. The grid for the finite-difference scheme is shown in Fig. (2).

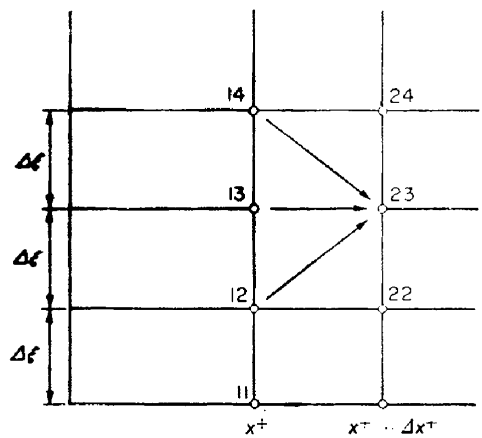

FIG. 2. Grid for the finite difference scheme.

From Fig. 2 it can be seen that all the values of temperature can be calculated for $T_{(x^++\Delta x^+)}$ from $T_{(x^+)}$, except for two end values, one at the wall and the second at $\xi = \xi_{max}$. The latter difficulty was met by selecting ξ_{max} so high that T there was zero. The value of T at the wall was obtained by adding the wall temperature gradient $\times \Delta\xi$ to the value of the temperature at $(x^+ + \Delta x^+, \Delta\xi)$.

To obtain the numerical results in solving equation (15), the complete calculations were made for an arbitrary value $(\partial T/\partial \xi)_{\xi=0} = -40$, and later the results were made dimensionless. The number -40 has no particular significance.

It was not possible to start the solution at $x^+ = 0$ because the temperature at all values of ξ at $x^+ = 0$ was zero. This difficulty was overcome by starting the solution at a very low value of x^+ called x_1^+. The starting value of x_1^+ for $(\partial T/\partial \xi)_{\xi=0} = -40$ was selected such that the temperature at all values of ξ was zero except at the surface. This first-assumed value of the surface temperature T_{w_1} was obtained from the approximate analysis of Smith and Shah [4]. The final results, when checked by integrating the heat flux, showed that they were not affected by any error in this assumption. The starting values for various Prandtl numbers are given in Table 1.

Table 1

Pr	T_{w_1}	x^+_1	$\Delta\xi$
0·7	19·30	0·050	0·5
1·0	19·41	0·025	0·5
7·0	38·50	0·004	1·0

The solution by the method of E. Schmidt is stable only if the condition

$$\frac{\Delta x^+}{f(\xi)(\Delta\xi)^2} < \tfrac{1}{2} \tag{17}$$

is fulfilled throughout. Since it was decided to solve the equation up to $x^+ = 10^6$, the interval Δx^+ was increased gradually as shown in Tables 2–4. With the increase in the interval Δx^+, it became necessary to increase $\Delta\xi$ near the surface (low values of ξ) to fulfill the above condition. As the values of $f(\xi)$ increase with ξ, the interval $\Delta\xi$ was gradually reduced for the calculations away from the surface. Near the surface, the temperature profile gradually became linear and therefore this increase in $\Delta\xi$ near the wall, with the increase in the interval Δx^+ did not impair the accuracy of the final results.

Table 2. *Steps chosen for* $Pr = 0{\cdot}7$

Interval x^+	Δx^+	$\Delta\xi$	Max. $\Delta\xi$
0 –1	0·05	0·5	0·5
1 –10	0·1	0·5	0·5
10 –10^2	1	0·5	1
10^2–10^3	1	0·5	1
10^4–10^5	50	0·5	4
10^5–10^6	500	0·5	8

Table 3. *Steps chosen for* $Pr = 1$

Interval x^+	Δx^+	$\Delta\xi$	Max. $\Delta\xi$
0 –1	0·025	0·5	0·5
1 –10	0·5	1	1
10 –10^2	1	1	1
10^2–10^3	4	1	2
10^3–10^4	20	1	4
10^4–10^5	25	1	4
10^5–10^6	250	1	8

Table 4. *Steps chosen for* $Pr = 7$

Interval x^+	Δx^+	$\Delta\xi$	Max. $\Delta\xi$
0 –1	0·005	1	1
1 –10	0·05	1	2
10 –10^2	0·5	1	4
10^2–10^3	1	2	6
10^3–10^4	5	2	8
10^4–10^5	25	2	14

FORM OF PRESENTATION OF THE RESULTS

The numbers resulting from the computations are temperature differences T in ξ, x^+ coordinates, with $(\partial T/\partial \xi)_{\xi=0} = -40$. Heat-transfer coefficients may be computed conveniently from such results in terms of the "Spalding function" $St/\sqrt{(c_f/2)}$, by the relation, deduced from equations (4), (8) and (12),

$$\frac{St}{\sqrt{(c_f/2)}} = -\frac{(\partial T/\partial \xi)_{\xi=0}}{T_w}. \tag{18}$$

Thus $[St/\sqrt{(c_f/2)}]\,(x^+)$ can be presented. Note, further, that

$$\frac{St}{\sqrt{(c_f/2)}} = \frac{\dot{q}''_w}{\rho C_p u T_w \sqrt{(c_f/2)}}. \tag{19}$$

Assuming that $c_f/2(x)$ and $u_1(x)$ are known for a specific example, $x^+(x)$ may be computed from equation (3). Hence, from $[St/\sqrt{(c_f/2)}](x^+)$, $St(x)$ may be computed. Then, knowing the constant value of $\dot{q}_w''/\rho C_p u_1 \sqrt{(c_f/2)}$, T_w may be found. Treatment for arbitrary $\dot{q}_w''$ is shown in a later section.

For the temperature within the boundary layer, it will then suffice if $\theta(\xi, x^+)$ be presented.

A practicable presentation is thus of $[St/\sqrt{(c_f/2)}](x^+)$ and $\theta(\xi, x^+)$.

CHECK BY HEAT BALANCE

An integral form of equation (15) may be derived by integrating it along the ξ co-ordinate. Thus

$$\frac{d}{dx^+}\int_0^\infty Tu^+\alpha^+\, d\xi = -\left(\frac{\partial T}{\partial \xi}\right)_{\xi=0}. \quad (20)$$

In our solution $(\partial T/\partial \xi)_{\xi=0} =$ const. and therefore

$$\int_0^\infty Tu^+\alpha^+\, d\xi = -\left(\frac{\partial T}{\partial \xi}\right)_{\xi=0} x^+. \quad (21)$$

Therefore, from equations (18) and (21) we have

$$\int_0^\infty \theta u^+\alpha^+\, d\xi = \frac{St}{\sqrt{(c_f/2)}}\, x^+. \quad (22)$$

The temperature profiles obtained in the numerical solution were integrated at various sections and the integrated heat fluxes were compared with the Spalding function and x^+ as shown in equation (22). Comparisons for various Prandtl numbers are tabulated in Tables 5–7. The results were found reasonably satisfactory.

Table 5. Integrated check of the results for Pr = 0·7

x^+	$\frac{St}{\sqrt{(c_f/2)}}x^+$	$\int_0^\infty \theta u^+\alpha^+\, d\xi$	% difference
10	3·810	3·838	+0·72
10^2	17·97	18·07	+0·54
10^3	97·15	97·21	+0·05
10^4	653·6	652·4	−0·18
10^5	4952	4876	−1·54
10^6	39 660	39 780	+0·30

Table 6. Integrated check of the results for Pr = 1

x^+	$\frac{St}{\sqrt{(c_f/2)}}x^+$	$\int_0^\infty \theta u^+\alpha^+\, d\xi$	% difference
10	2·965	3·010	+1·51
10^2	14·18	14·22	+0·30
10^3	77·28	77·48	0·27
10^4	544·1	546·6	+0·44
10^5	4293	4298	+0·11
10^6	35 320	35 190	−0·37

Table 7. Integrated check of the results for Pr = 7

x^+	$\frac{St}{\sqrt{(c_f/2)}}x^+$	$\int_0^\infty \theta u^+\alpha^+\, d\xi$	% difference
10	0·8185	0·7994	−2·33
10^2	3·844	3·856	+0·31
10^3	20·75	20·57	−0·87
10^4	171·3	171·2	−0·06
10^5	1571	1579	+0·52

DEPENDENCE OF THE SPALDING FUNCTION AND θ ON THE PRANDTL NUMBER

The results of the calculations in terms of the temperature profiles are shown in Figs. 3–5. The values of the Spalding function for various values of x^+ and for Prandtl numbers 0·7, 1 and 7 are tabulated in Tables 8–10 and plotted in Fig. 6.

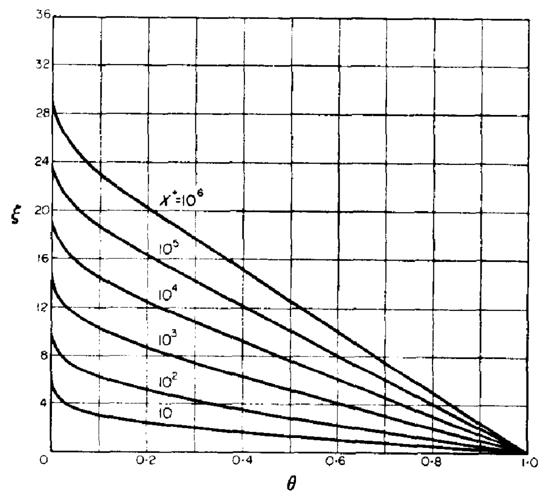

FIG. 3. Temperature profiles for $Pr = 0·7$.

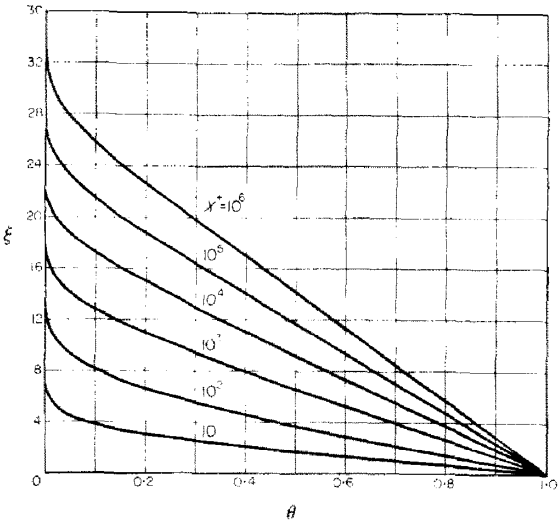

FIG. 4. Temperature profiles for $Pr = 1$.

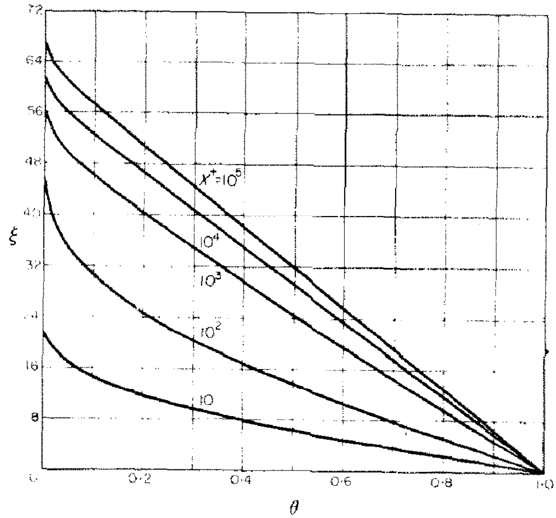

FIG. 5. Temperature profiles for $Pr = 7$.

Table 8. The Spalding function $St/\sqrt{(c_f/2)}$ for $Pr = 0{\cdot}7$

x^+	$St/\sqrt{(c_f/2)}$	x^+	$St/\sqrt{(c_f/2)}$
1	0·801678	1×10^3	0·097158
2	0·642647	2×10^3	0·084421
3	0·563886	3×10^3	0·078568
4	0·513691	4×10^3	0·074916
5	0·477754	5×10^3	0·072325
6	0·450215	6×10^3	0·070346
7	0·428144	7×10^3	0·068760
8	0·409884	8×10^3	0·067446
9	0·394414	9×10^3	0·066330
10	0·381063	1×10^4	0·065363
20	0·300255	2×10^4	0·059528
30	0·263463	3×10^4	0·056654
40	0·240197	4×10^4	0·054776
50	0·223636	5×10^4	0·053401
60	0·211018	6×10^4	0·052325
70	0·200964	7×10^4	0·051448
80	0·192695	8×10^4	0·050711
90	0·185731	9×10^4	0·050077
1×10^2	0·179756	1×10^5	0·049523
2×10^2	0·146030	2×10^5	0·045895
3×10^2	0·130328	3×10^5	0·044175
4×10^2	0·120738	4×10^5	0·043020
5×10^2	0·114091	5×10^5	0·042159
6×10^2	0·109125	6×10^5	0·041476
7×10^2	0·105228	7×10^5	0·040913
8×10^2	0·102059	8×10^5	0·040436
9×10^2	0·099413	9×10^5	0·040023
		10^6	0·039660

Table 9. The Spalding function $St/\sqrt{(c_f/2)}$ for $Pr = 1$

x^+	$St/\sqrt{(c_f/2)}$	x^+	$St/\sqrt{(c_f/2)}$
1	0·61424	1×10^3	0·07728
2	0·49137	2×10^3	0·067892
3	0·43359	3×10^3	0·063685
4	0·39639	4×10^3	0·061115
5	0·36957	5×10^3	0·059300
6	0·34889	6×10^3	0·057914
7	0·33224	7×10^3	0·056803
8	0·31841	8×10^3	0·055880
9	0·30667	9×10^3	0·055095
10	0·29650	1×10^4	0·054413
20	0·23723	2×10^4	0·050335
30	0·20810	3×10^4	0·048244
40	0·18965	4×10^4	0·046863
50	0·17709	5×10^4	0·045843
60	0·16652	6×10^4	0·045042
70	0·15856	7×10^4	0·044385
80	0·15201	8×10^4	0·043830
90	0·14650	9×10^4	0·043351
1×10^2	0·14177	1×10^5	0·042931
2×10^2	0·11513	2×10^5	0·040295
3×10^2	0·10281	3×10^5	0·038927
4×10^2	0·09510	4×10^5	0·038007
5×10^2	0·08998	5×10^5	0·037319
6×10^2	0·08621	6×10^5	0·036774
7×10^2	0·08327	7×10^5	0·036324
8×10^2	0·08091	8×10^5	0·035943
9×10^2	0·07894	9×10^5	0·035612
		10^6	0·035321

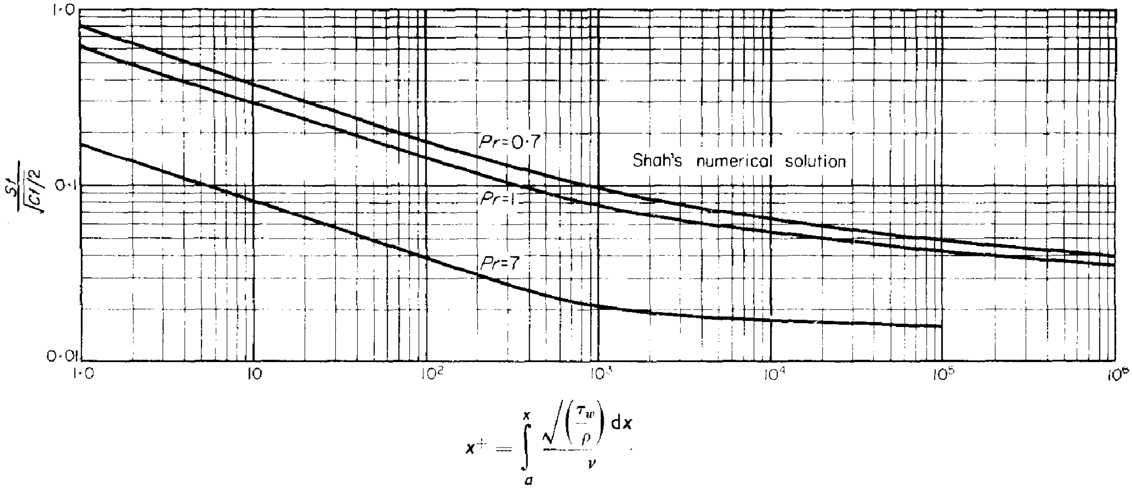

FIG. 6. The Spalding function $[St/\sqrt{(c_f/2)}]$ (x^+) for $Pr = 0{\cdot}7$, 1 and 7.

Table 10. The Spalding function $St/\sqrt{(c_f/2)}$ for $Pr = 7$

x^+	$\frac{St}{\sqrt{(c_f/2)}}$	x^+	$\frac{St}{\sqrt{(c_f/2)}}$
1	0·172205	1×10^3	0·020749
2	0·137977	2×10^3	0·018916
3	0·121054	3×10^3	0·018277
4	0·110283	4×10^3	0·017932
5	0·102576	5×10^3	0·017705
6	0·096672	6×10^3	0·017537
7	0·091940	7×10^3	0·017406
8	0·088025	8×10^3	0·017297
9	0·084709	9×10^3	0·017206
10	0·081847	1×10^4	0·017127
20	0·064768	2×10^4	0·016646
30	0·056777	3×10^3	0·016395
40	0·051701	4×10^4	0·016225
50	0·048080	5×10^4	0·016096
60	0·045315	6×10^4	0·015993
70	0·043107	7×10^4	0·015907
80	0·041289	8×10^4	0·015834
90	0·039754	9×10^4	0·015770
		10^5	0·015713
1×10^2	0·038435		
2×10^2	0·030841		
3×10^2	0·027396		
4×10^2	0·025336		
5×10^2	0·023950		
6×10^2	0·022950		
7×10^2	0·022196		
8×10^2	0·021607		
9×10^2	0·021135		

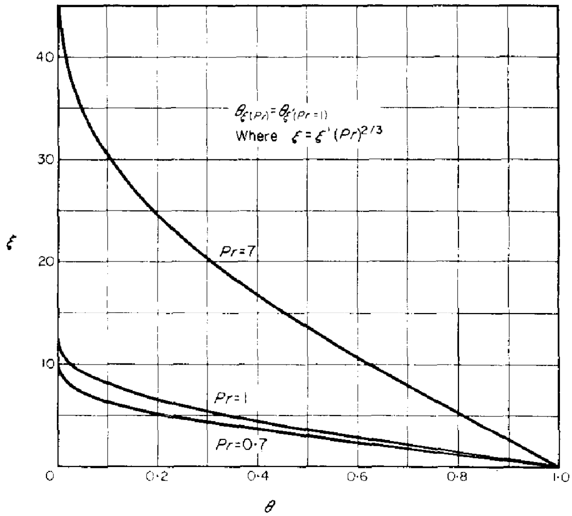

FIG. 7. Temperature profiles for various Prandtl numbers ($x^+ = 100$) (laminar region).

Having performed these calculations and checks, an effort was made to find the dependence on Prandtl number of the Spalding function and the temperature profile in the form

$$\frac{St}{\sqrt{(c_f/2)}} = \frac{St}{[\sqrt{(c_f/2)}]_{(Pr=1)}} \times Pr^n. \quad (23)$$

Results for $Pr = 0{\cdot}7$, 1 and 7 showed that in the laminar region (up to $x^+ = 1000$) the value of n remains constant and is equal to $-2/3$; but, for higher x^+, the value of n when obtained from the results for $Pr = 0{\cdot}7$ and 1, changes from $-2/3$ at $x^+ = 1000$ to $-0{\cdot}4$ at $x^+ = 10^5$ and $-0{\cdot}325$ at $x^+ = 10^6$, whereas from the results for $Pr = 7$, the value of n decreases from $-2/3$ at $x^+ = 1000$ to $-0{\cdot}516$ at $x^+ = 10^5$.

In the laminar region ($x^+ < 1000$) the temperature profile for any Prandtl number can be obtained from

$$\theta_{(\xi)} = \theta_{(\xi')\,(Pr=1)} \tag{24}$$

where

$$\xi = \xi' \times Pr^{2/3}. \tag{25}$$

Temperature profiles for various Prandtl numbers and for $x^+ = 100$ are shown in Fig. 7.

APPLICATION OF THE SOLUTIONS TO THE CASE OF ARBITRARY HEAT FLUX AT THE WALL

The working equation by which the numerical solutions given in this paper may be applied is equation (32) below. A full statement of its development is given in sections A and B following.

A. *Determination of temperature at the wall*

The numerical solutions given in the previous sections have the boundary condition $(\partial T/\partial \xi)_0 = \text{const.}\ x^+ > 0$. From the definitions of the variables in equations (6–8) and earlier, together with the velocity-law equation (12), this boundary condition is the same as $[\dot{q}''_w/\rho u_1 C_p \sqrt{(c_f/2)}] = \text{const.}$ That is, the heat flux parameter is constant.

This boundary condition is highly particular: usually problems will be met in the form of a specified distribution of $\dot{q}''_w$, with $T_w(x)$ the quantity desired to be known.

However, the treatment of the problem of arbitrary $\dot{q}''_w$ will be most easily perceived after a recapitulation of the method of determination of $T_w(x)$ for constant $\dot{q}''_w/\rho u_1 C_p \sqrt{(c_f/2)}$.

Fig. 6 gives values of the Spalding function $[St/\sqrt{(c_f/2)}]\,(Pr, x^+)$. If $St/\sqrt{(c_p/2)}$ be known at a given x, then T_w is given by

$$T_w = \frac{1}{St/\sqrt{(c_f/2)}} \cdot \left[\frac{\dot{q}''_w}{\rho u_1 C_p \sqrt{(c_f/2)}}\right]. \tag{26}$$

In Fig. 6, heat injection starts at $x = a$. The problem is, then, to calcuiate the temperature $T_{w\,(b)}$ at $x = b$, consequent on constant $\dot{q}''_w/\rho u_1 C_p \sqrt{(c_f/2)}$ between $x = a$ and $x = b$.

Define a new variable $x_{a,b}$ by the equation

$$x^+_{a,b} = \int_a^b \frac{\sqrt{(\tau_w/\rho)}}{\nu}\,\mathrm{d}x. \tag{27}$$

Then $x^+_{a,b}$ is easily calculable, knowing $\tau_w(x)$, ρ and ν. From Fig. 6, $[St/\sqrt{(c_f/2)}]_{a,b}$ may be read off, for $x^+ = x^+_{a,b}$. This quantity is the value of $[St/\sqrt{(c_f/2)}]$ at $x = b$ consequent on constant $[\dot{q}''_w/\rho u_1 C_p \sqrt{(c_f/2)}]$ from $x = a$ to $x = b$. Then at $x = b$ the difference $T_{w\,(b)}$ between wall temperature and mainstream is given by

$$T_{w\,(b)} = \frac{1}{[St/\sqrt{(c_f/2)}]_{a,b}} \cdot \left[\frac{\dot{q}''_w}{\rho u_1 C_p \sqrt{(c_f/2)}}\right]_a. \tag{28}$$

It must be particularly noted that equation (28) holds for *constant* $\dot{q}''_w/\rho u_1 C_p \sqrt{(c_f/2)}$ between $x = a$ and $x = b$. However, in the general problem, $\dot{q}''_w/\rho u_1 C_p \sqrt{(c_f/2)}$ will be a function of x. Knowing $\dot{q}''_w(x)$, $u_1(x)$, $c_f(x)$, the dependence of $\dot{q}''_w/\rho u_1 C_p \sqrt{(c_f/2)}$ on x may be computed. The temperature at the wall at $x = b$ is then given by the Stieltjes integral

$$T_{w\,(b)} = \int_0^b \frac{1}{[St/\sqrt{(C_f/2)}]_{a,b}}\,\mathrm{d}\left[\frac{\dot{q}''_w}{\rho u_1 C_p \sqrt{(c_f/2)}}\right]_a. \tag{29}$$

Equation (29) is the working equation for the determination of wall temperature. In equation (29), $[St/\sqrt{(c_f/2)}]_{a,b}$ is the value of $St/\sqrt{(c_f/2)}$ at b, consequent on *constant* $\dot{q}''_w/\rho u_1 C_p \sqrt{(c_f/2)}$ between $x = a$ and $x = b$, $\mathrm{d}[\dot{q}''_w/\rho u_1 C_p \sqrt{(c_f/2)}]_a$ is the increment of heat-flux parameter for the increment $\mathrm{d}x$ containing $x = a$. It may be more convenient for computation if equation (29) is rewritten:

$$T_{w\,(b)} = \int_0^b \frac{1}{[St/\sqrt{(c_f/2)}]_{a,b}}\,\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\dot{q}''_w}{\rho u_1 C_p \sqrt{(c_f/2)}}\right]_a \mathrm{d}x. \tag{30}$$

In evaluating equation (30), however, it must be noted that if there is a step in $\dot{q}''_w/\rho u_1 C_p \sqrt{(c_f/2)}$, this step contributes an increment of $T_{w\,(b)}$ given by

$$\Delta T_{w(b)} = \frac{1}{[St/\sqrt{(c_f/2)}]_{a,b}} \Delta \left[\frac{\dot{q}''_w}{\rho u_1 C_p \sqrt{(c_f/2)}} \right]_a . \quad (31)$$

In equation (31), the symbol Δ signifies a finite increment.

B. *Determination of temperature within the boundary layer*

The previous section showed the method of computing wall temperature consequent on an arbitrary distribution of wall heat flux. A further problem is, however, the determination of temperature within the boundary layer. Again the method will be seen more easily if the computation for constant $\dot{q}''_w/\rho u_1 C_p \sqrt{(c_f/2)}$ is first considered.

Temperatures within the boundary layer have been presented in Figs. (3–5) as $\theta(x^+, \xi, Pr)$. θ, defined as in the Nomenclature, is in fact temperature on a scale such that mainstream temperature is zero and wall temperature is unity. The temperature at a point in the boundary layer is defined if T_w and θ be known. The relation is: $T = \theta T_w$.

Concentrating our attention on the case of constant $\dot{q}''_w/\rho u_1 C_p \sqrt{(c_f/2)}$, and temperature difference between wall and mainstream at $x = b$, called $T_{w(b)}$, may be found by the method of the previous section. The problem now is to find the fluid temperature at point c with co-ordinate x, y. Of the parameters in Figs. 3–5, x^+ and Pr are already known. To determine θ and thus permit the determination of temperature at y, the dimensionless co-ordinate ξ must be determined for the point c. ξ is defined by equation (8), and equations (5–8) and (12–14) permit the function $\xi(y^+, Pr)$ to be computed. Results of this computation are shown in Fig. 8 and Table 11. Using Fig. 8, ξ may therefore be found, after y^+ has been determined from equation (4). Thence θ may be read off Fig. 3, 4

Table 11. $\xi(y^+, Pr)$ *and* $y^+(u^+)$

u^+	y^+	ξ, $Pr = 0·7$	ξ, $Pr = 7·0$	u^+	y^+	ξ, $Pr = 0·7$	ξ, $Pr = 7·0$
0	0	0	0	19	219·0	15·90	59·35
1	1	0·7	6·999	20	328·6	16·89	60·36
2	2	1·4	13·99	21	494·6	17·89	61·36
3	3·003	2·101	20·88	22	745·6	18·88	62·37
4	4·013	2·803	27·50	23	1124	19·87	63·37
5	5·042	3·509	33·48	24	1696	20·87	64·37
6	6·115	4·223	38·48	25	2557	21·87	65·38
7	7·274	4·953	42·35	26	3854	22·87	66·38
8	8·590	5·702	45·25	27	5806	23·87	67·38
9	10·18	6·496	47·44	28	8743	24·87	68·38
10	12·23	7·322	49·18	29	13 160	25·87	69·38
11	15·04	8·189	50·62	30	19 800	26·87	70·38
12	19·06	9·092	51·90	31	29 780	27·87	71·38
13	24·98	10·02	53·07	32	44 780	28·87	72·38
14	33·88	10·98	54·18	33	67 310	29·87	73·38
15	47·36	11·95	55·25	34	101 200	30·87	74·38
16	67·88	12·92	56·29	35	152 100	31·87	75·38
17	99·13	13·91	57·32	36	228 500	32·87	76·38
18	146·7	14·90	58·34	37	343 400	33·87	77·38

(*a*) For $Pr = 1$, $\xi = u^+$.
(*b*) For $y^+ > 300$ the following relations hold.
$y^+ = 0·3255 \exp (0·4098\, \xi)$ $(Pr = 0·7)$,
$y^+ = 0·09177 \exp (0·4093\, \xi)$ $(Pr = 1·0)$
$y^+ = 6·317 \times 10^{-9} (\exp 0·4088\, \xi)$ $(Pr = 7·0)$.

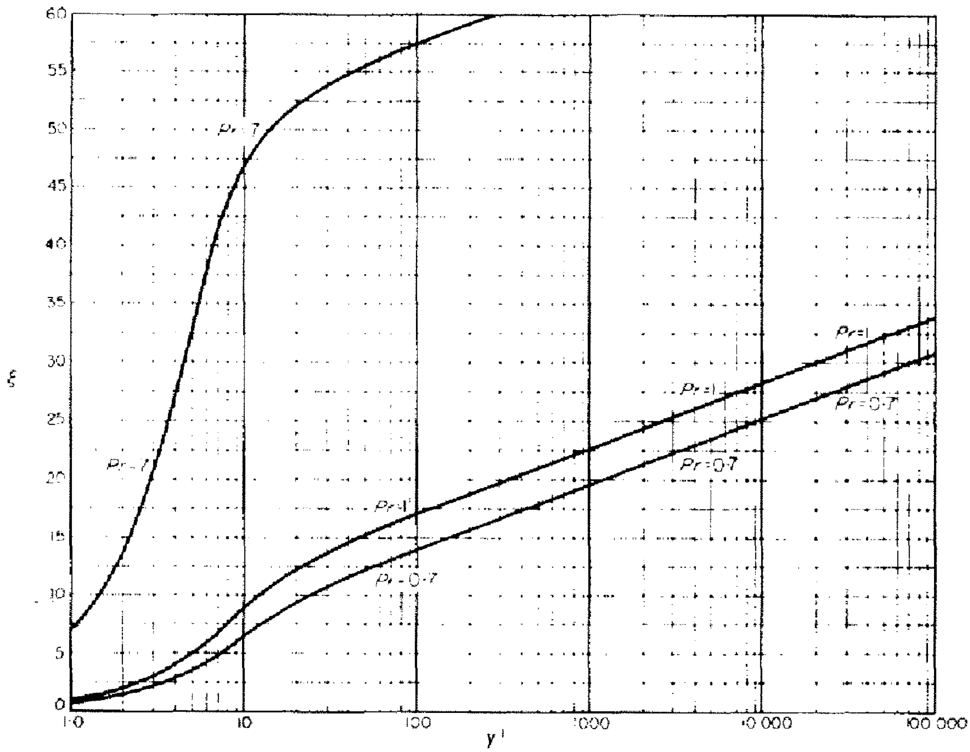

FIG. 8. Values of $\xi(y^+, Pr)$.

or 5. The difference of temperature between point c and the mainstream is then

$$T_{c(b)} = \frac{\theta_{a,b,c}}{[St/\sqrt{(c_f/2)}]_{a,b}}\left[\frac{\dot{q}''_w}{\rho u_1 C_p \sqrt{(c_f/2)}}\right]_a. \quad (32)$$

In equation (32), $\theta_{a,b,c}$ is the value of θ at $x^+ = x^+_{a,b}$ and the appropriate ξ. $[St/\sqrt{(c_f/2)}]_{a,b}$ is the value read off Fig. 6 at $x^+ = x^+_{a,b}$ and $\dot{q}''_w/\rho u_1 C_p \sqrt{(c_f/2)}$ is the constant value of the heat flux parameter between $x = a$ and $x = b$.

When there is an arbitrary distribution $\dot{q}''_w(x)$, the distribution of the heat-flux parameter may be computed as described in the previous section of the paper, and the value of the difference in temperature between point c and the mainstream may be computed by the Stieltjes integral:

$$T_{c(b)} = \int_0^b \frac{\theta_{a,b,c}}{[St/\sqrt{(c_f/2)}]_{a,b}}\, \mathrm{d}\left[\frac{\dot{q}''_w}{\rho u_1 C_p \sqrt{(c_f/2)}}\right]_a. \quad (33)$$

Equation (33) is the general equation by which the results of this paper may be applied, since it comprehends equation (29) in that, at the wall, $\theta_{a,b,c}$ is unity. Again it may be more convenient for computation if the equation is rewritten

$$T_{c(b)} = \int_0^b \frac{\theta_{a,b,c}}{[St/\sqrt{(c_f/2)}]_{a,b}} \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\dot{q}''_w}{\rho u_1 C_p \sqrt{(c_f/2)}}\right]_a \mathrm{d}x. \quad (34)$$

In the evaluation of equation (34) care must be taken in allowing for steps in $\dot{q}''_w/\rho u_1 C_p \sqrt{(c_f/2)}$, as was pointed out after equation (30).

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Résumé—Une solution numérique de l'équation aux dérivées partielles du transport de chaleur dans la couche limite turbulente incompressible a été obtenue pour $(\dot{q}''_w/\rho C_p u_1)1/\sqrt{(c_f/2)}$ constant et pour les nombres de Prandtl 0,7, 1 et 7. La loi des vitesses de Spalding a été adoptée et on a utilisé la méthode d'intégration de Schmidt. Les distributions de températures dans la couche limite sont données pour des x^+ allant jusqu'a 10^6, en même temps que la fonction de Spalding $St/\sqrt{(c_f/2)}$. On donne une méthode pour l'application de ces solutions au cas d'une distribution arbitraire du flux thermique à la paroi.

Zusammenfassung—Für den Wärmeübergang in der inkompressiblen turbulenten Granzschicht wurde die numerische Lösung der partiellen Differentialgleichung bei konstantem $\dot{q}''_w/\rho C_p u_1 (c_f/2)^{-1/2}$ erhalten für Prandtlzahlen 0,7, 1 und 7. Das Spalding-Gesetz der Grenzschichtgeschwindigkeit ist vorausgesetzt, die Integration wurde nach der Schmidt-Methode durchgeführt. Die Grenzschichttemperaturverteilungen bis $x^+ = 10^6$ sind zusammen mit der "Spalding-Funktion" $St(c_f/2)^{-1/2}$ angegeben. Für beliebige Verteilung des Wärmeflusses an der Wand ist eine Lösungsmethode beschrieben.

Аннотация—Получено численное решение дифференциального уравнения в частных производных для переноса тепла в турбулентном пограничном слое несжимаемой жидкости при постоянной величине $\dot{q}''_w/\rho C_p u_1(c_f/2)^{-1/2}$ и значениях критерия Прандтля, равных 0,7, 1 и 7. Принят закон Сполдинга для скорости пограничного слоя и использован метод интегрирования Шмидта. Приводятся значения распределения температур пограничного слоя до $x^+ = 10^6$ вместе с «функцией Сполдинга» $St(c_f/2)^{-1/2}$. Дается метод применения решений к случаю произвольного распределения теплового потока на стенке.